CS 3913 Final Examination

Friday August 11, 2006, 9:00 am.

The time limit is three hours. No aids allowed. J. D. Horton

All questions are of equal value (10). Marks total 100.

On the front of each answer booklet, there is a list of the questions. Please circle the number of any question answered in the booklet. If you do not answer a question, then put an X thru the number of the question not answered on your first booklet, and I will give you one mark.

1. Some students, A, B, ..., X are looking for co-op jobs. Several companies, d, e, ..., y are looking to hire students. Each student has a preference list of the jobs they would like, and the companies have preference lists of the students they would like to hire. However they would like the assignment of jobs to students to be stable, that is, there is no situation where student X works for company z, student Y works for company w, and company z prefers student Y to X, and student Y prefers company z to company w, because in that case Y could quit their job at w and work for z who would refuse to hire X. However no knows anyone else's preference lists.

   Every day, all the companies can email an offer to one student of their choice. At the end of the day, each student can reject any offer sent to them during that day, or any previous day.

   (a) Explain how the Gale-Shapley can be used to find a stable matching if the number of companies and the number of students are equal.

   (b) Does the algorithm still work if there are more companies than students? Explain your answer.

   (c) Does the algorithm still work if there are more students than companies? Explain your answer.

2. Put the following functions into asymptotic increasing order.

$$2^n, (\log n)^5, 100n, 10^{20}, n \log n, n^2, n!, n^{\log n}$$

3. Let $f(1) = a$ and $f(2n) = 2f(n) + cn$. Prove that $f(2^k) = \Theta(k2^k)$.

4. Give an example of a graph for which Depth First Search visits the nodes in the order 1, 2, 3, 4, 5, 6, 7, while Breadth First Search visits the nodes in the order 1, 2, 7, 3, 6, 4, 5.

5. Below is a list of edges and weights of a weighted undirected graph. Draw the graph and find the shortest path from A to B. Specify the order of the vertices in which Dijkstra's algorithm determines the distance from A.

   The edges of the graph are:

   DF-1, BF-1, ED-2, AC-2, BD-3, EA-3, CD-4, EF-5 BC-8.

6. We are given n jobs, with known running times of $t_1, t_2, \ldots, t_n$. We have a single machine, and must run all the jobs on this machine. We want to minimize the average completion time. In what order should we schedule the jobs? Hint: use a greedy algorithm. Prove that your algorithm solves the problem, in that the average completion in smaller in your schedule than any other schedule.

   For example, if the length of the jobs are 2, 1 and 3, and the jobs are run in this order, then the finish times of the three jobs are: 2, 2+1 = 3, and 2+1+3 = 6 for an average of 3 and 2/3. However a better solution is to order the jobs so that the job of length 1 runs first, then the job of length 2, and finally the job of length 3. Then the finish times are 1, 1+2 = 3, and 1+2+3 = 6, for an average of 3 and 1/3.

7. We have been given the id numbers in random order of all the students registered to vote in the next student election. How can we check to see if there are any duplicates in the list? Give an efficent algorithm to do this. What is the asymptotic time-complexity of your algorithm?

8. You are managing the construction of billboards on a highway of length M kilometers. The possible sites are at points $x_1 < x_2 < \ldots < x_n$ kilometers from the start of the highway. The value of the site at site i, that is at location x_i, is v_i. Thus site i is at position x_i and has value v_i Regulations require that billboards be more than 5 kilometers apart. You want to maximize the value of the set of sites at which billboards are constructed.

   Prove that the following two greedy algorithms do not solve the problem.

(a) Choose the first site, and then until we have reached the end of the highway, choose the first site that is more than 5 kilometers away from the previously chosen site.

(b) Choose the site with the highest value, then the next site with the highest value that is not within 5 kilometers of a previously chosen site, etc.

(c) Devise a dynamic programming algorithm to find the value of the best set of sites at which to place billboards in the previous question.

9. (a) The Independent Set Problem is defined as follows:

   Instance: A graph $G = (V, E)$ and a positive integer bound b.

   Question: Does G contain a set of b vertices such that no two of these vertices are connected by an edge?

   Prove that the independent set problem is in NP.

   (b) Assume that the Jabberwock Problem is in P, and the Worple Problem is NP-complete. If you reduce the Jabberwock problem to the Worple problem, does this prove that $P = NP$? Give a short explanation.

10. The Hamilton Cycle problem (HC) asks whether an undirected $G = (V, E)$ has a hamilton cycle. The Directed Hamilton Cycle problem (DHC) asks whether a directed graph $H = (N, A)$ has a hamilton cycle. Give a reduction from HC to DHC. In other words, prove

$$HC \leq_P DHC$$